56—4

Proceedings of the American Academy of Arts and Sciences.

Vol. 56. No. 4.— March, 1921.

MOTION ON A SURFACE FOR ANY POSITIONAL FIELD OF FORCE.

By Joseph Lipka.

(Continued from page 3 of cover.)

VOLUME 56.

Proceedings of the American Academy of Arts and Sciences.

Vol. 56. No. 4.— March, 1921.

MOTION ON A SURFACE FOR ANY POSITIONAL FIELD OF FORCE.

By Joseph Lipka.

MOTION ON A SURFACE FOR ANY POSITIONAL FIELD OF FORCE.

By Joseph Lipka.

Received Nov. 8, 1920. Presented Nov. 8, 1920.

§ 1. Introduction.

The first part of the paper presents a study of the geometric properties of the system of trajectories generated by the motion of a particle on any constraining surface (spread of two dimensions) under any positional forces. The complete characteristic properties are derived. [1] Starting at any point on the surface in a given direction and with a given speed, a unique trajectory is generated, and the complete set of trajectories forms a triply infinite system of curves corresponding uniquely to a given field of force.

Through a given point O and in a given direction, there pass ∞^1 trajectories. We associate with these trajectories the ∞^1 curves obtained by orthogonal projection into the tangent plane to the surface at O. The first two properties derived deal with the bicircular quartic which is the locus of the foci of the osculating parabolas of the associated system.

A second set of geometric properties is derived by considering the ∞^2 trajectories through a point O on the surface, and the directions through O in which the trajectories hyperosculate (have 4-point contact with) their corresponding geodesic circles of curvature. It is found that the hyperosculation property holds for only one trajectory in each direction, and that the corresponding locus of the centers of geodesic curvature is a conic.

A final set of properties is then derived showing the relations existing at a point between the geodesic curvatures of the trajectories and the lines of force, and any isothermal net of curves on the surface. It is shown that the entire five properties are characteristic of the system

---

1 For a study of the corresponding problem in a plane and in ordinary 3-space, see Edward Kasner, "*The trajectories of dynamics,*" Trans. Am. Math. Soc., **7** (1906), pp. 401–424; also "*Dynamical Trajectories: the motion of a particle in an arbitrary field of force,*" Trans. Am. Math. Soc., **8** (1907), pp. 135–158. The results of these 2 papers are summarized in Professor Kasner's Princeton Colloqium Lectures on the differential geometric aspects of dynamics, chapt. I.

of trajectories, and that any triply infinite system of curves on a surface possessing these five properties may be considered as generated by the motion of a particle in a unique field of force.

We next point out how an additional property serves to characterize the motion when the field of force is conservative.

Another part of this paper presents briefly an analogous study for certain other classes of triply infinite systems of curves on a surface, in particular, brachistochrones, catenaries and velocity curves in a conservative field of force. For all such systems characteristic properties differing but slightly from those for trajectories are derived.

§ 2. Differential Equation of the Trajectories.

If we choose an isothermal net of curves as parameter curves on the surface

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \tag{1}$$

the element of arc length may be written

$$ds^2 = \mu(u, v)\,[du^2 + dv^2]. \tag{2}$$

The motion of a particle on the surface may be most simply expressed by the Lagrangian equations [2]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \phi, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \psi \tag{3}$$

where T is the kinetic energy

$$2T = \mu(\dot{u}^2 + \dot{v}^2), \tag{4}$$

and φ and ψ are the components of the force given as functions of the coördinates u, v.[3] Introducing the value of T in (3), we get the explicit equations of motion

$$\left\{\begin{aligned} \ddot{u} &= \frac{1}{\mu}\left[\phi - \frac{1}{2}(\mu_u \dot{u}^2 + 2\mu_v \dot{u}\dot{v} - \mu_u \dot{v}^2)\right] \\ \ddot{v} &= \frac{1}{\mu}\left[\psi + \frac{1}{2}(\mu_v \dot{u}^2 - 2\mu_u \dot{u}\dot{v} - \mu_v \dot{v}^2)\right], \end{aligned}\right. \tag{5}$$

2 See E. T. Whittaker, *Analytical Dynamics*, p. 39.

3 Throughout the paper, dots refer to derivatives with respect to t (time), primes refer to total derivatives with respect to u, and literal subscripts to partial derivatives.

or if, for convenience, we write

$$(6)\qquad \lambda = \frac{1}{2}\log\mu, \text{ and } \mu = e^{2\lambda},$$

equations (5) become

$$(7)\qquad \begin{cases} \ddot{u} = \dfrac{\phi}{e^{2\lambda}} - (\lambda_u\dot{u}^2 + 2\lambda_v\dot{u}\,\dot{v} - \lambda_u\dot{v}^2) \\[2ex] \ddot{v} = \dfrac{\psi}{e^{2\lambda}} + (\lambda_v\dot{u}^2 - 2\lambda_u\dot{u}\,\dot{v} - \lambda_v\dot{v}^2). \end{cases}$$

To get the differential equations of the trajectories we must eliminate the time from equations (7). We evidently have

$$v' = \frac{\dot{v}}{\dot{u}}, \quad v'' = \frac{\dot{u}\,\ddot{v} - \dot{v}\,\ddot{u}}{\dot{u}^3},$$

and hence

$$(8)\qquad v'' = \frac{1}{e^{2\lambda}\dot{u}^2}(\psi - \phi\, v') + (\lambda_v - \lambda_u\, v')\,(1 + v'^2).$$

Using the abbreviations

$$(9)\qquad \begin{cases} G \equiv v'' - (\lambda_v - \lambda_u\, v')\,(1 + v'^2), \\[1ex] G' \equiv \dfrac{dG}{du} \equiv v''' + v''\,(\lambda_u - 2\lambda_v\, v' + 3\lambda_u\, v'^2) \\[1ex] \qquad\qquad - (\lambda_{uv} + \lambda_{vv}\, v' - \lambda_{uu}\, v' - \lambda_{uv}\, v'^2)\,(1 + v'^2), \end{cases}$$

where $G = 0$ is the differential equation of the geodesics on the surface, we may write (8) in the form

$$(10)\qquad \dot{u}^2 = \frac{\psi - \phi v'}{e^{2\lambda}\, G}.$$

Differentiating (10) with respect to u and using (7), we get

$$(11)\quad (\psi - \phi v')\, G' = G\{(\psi_u + 2\lambda_v\phi) + (\psi_v - \phi_u + 2\lambda_v\psi - 2\lambda_u\phi)v' \\ - (\phi_v + 2\lambda_u\psi)v'^2 - 3\phi v''\},$$

as the differential equation of the trajectories. If we replace G and G' by their values from (9), we get a differential equation of the form

$$(12)\qquad v''' = P + Qv'' + Rv''^2$$

where

$$(13)\qquad \begin{cases} P = (\alpha_0 + \alpha_1 v' + \alpha_2 v'^2 + \alpha_3 v'^3 + \alpha_4 v'^4 + \alpha_5 v'^5)/(\psi - \phi v'), \\ Q = (\beta_0 + \beta_1 v' + \beta_2 v'^2 + \beta_3 v'^3)/(\psi - \phi v'), \\ R = -3\phi/(\psi - \phi v'), \end{cases}$$

the α's and β's being functions of ϕ, ψ, and λ, and hence of u and v. In deriving the geometric properties of the trajectories we shall use the equation in the form (11) almost exclusively.

The triply infinite system of curves represented by (11) is thus uniquely determined by the force components ϕ, ψ. Two different fields of force ϕ, ψ and $\bar{\phi}$, $\bar{\psi}$ cannot give rise to the same system of trajectories. For if the two systems

$$(14)\quad G' = G(a + bv' + cv'^2 + dv'') \text{ and } G' = G(\bar{a} + \bar{b}v' + \bar{c}v'^2 + \bar{d}v'')$$

coincide, we must have

$$(15)\qquad a = \bar{a},\ b = \bar{b},\ c = \bar{c},\ d = \bar{d}.$$

The last of these equations, by comparison with (12) and (13), gives

$$(16)\qquad \bar{\psi}/\psi = \bar{\phi}/\phi,$$

so that we may write

$$(17)\qquad \bar{\psi} = \alpha\psi,\ \bar{\phi} = \alpha\phi,$$

where α is some function of u, v; substituting these values in the first three equations, we find

$$(18)\qquad \alpha_u = 0,\ \alpha_v = 0,$$

and hence α is simply a constant, k. Therefore the forces are the same except for a constant factor, which corresponds merely to a change in the unit of force. Hence we may state

THEOREM 1. *The system of trajectories corresponding to a field of force completely defines that field.*

We further note that $G = 0$ satisfies equation (11), so that the geodesics form part of every system of trajectories, i.e. for every field of force. Indeed, from (8), we note that the speed of the particle describing a geodesic is infinite.

§ 3. The Associate System and the Focal Locus.

Consider the ∞^1 trajectories passing through a point O in a direction v', i.e. having a common initial element (u, v, v'). Project these curves orthogonally into the tangent plane at O.[4] We shall call the

[4] For our purpose it is merely necessary to project the elements up to the third order, v', v'', v'''.

resulting curves the associate system corresponding to the given initial element. Our first query is: what is the locus of the foci of the osculating parabolas of the ∞^1 curves of the associate system? To assist us in answering this, we shall choose the tangent plane as the XOY plane, the point O as the origin, and the OZ axis as the normal to the surface. We shall further choose the x- and y- axes as the tangent lines to our isothermal v- and u- parameter curves, respectively.

Here $z = 0$. The trajectory determined by (o, o, v', v'', v''') is associated with the curve determined by $\left(o, o, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right)$. The latter derivatives may be expressed in terms of the former by using the equations of the surface (1). Thus we have

$$(19)\quad x' = x_u + x_v v',\ x'' = x_{uu} + 2x_{uv}v' + x_{vv}v'^2 + x_v v'',$$
$$x''' = x_{uuu} + 3x_{uuv}v' + 3x_{uvv}v'^2 + x_{vvv}v'^3 + 3x_{uv}v'' + 3x_{vv}v'\,v'' + x_v v''',$$

with similar expressions for y', y'', y'''. Then,

$$\frac{dy}{dx} = \frac{y_u + y_v v'}{x_u + x_v v'}. \tag{20}$$

By our choice of isothermal parameters, the tangent of the angle between the initial element and the v- parameter curve is v', so that

$$x_v = 0,\ y_u = 0,\ x_u = y_v. \tag{21}$$

Differentiating (20) and using (21), we find

$$\left\{\begin{aligned} \frac{dy}{dx} &= v', \\ \frac{d^2y}{dx^2} &= \frac{1}{x_u}v'' + a, \\ \frac{d^3y}{dx^3} &= \frac{1}{x_u^2}v''' + b\,v'' + c, \end{aligned}\right. \tag{22}$$

where a, b, and c are functions of u, v, v' only.

Now the coördinates (α, β) of the focus of the osculating parabola to the associate element $\left(o, o, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right)$ are given by

$$
(23)\left\{\begin{aligned}
2\alpha &= \frac{-3\dfrac{d^2y}{dx^2}\left\{\dfrac{d^3y}{dx^3}\left[\left(\dfrac{dy}{dx}\right)^2-1\right]+2\dfrac{dy}{dx}\left[3\left(\dfrac{d^2y}{dx^2}\right)^2-\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}\right]\right\}}{\left(\dfrac{d^3y}{dx^3}\right)^2+\left[3\left(\dfrac{d^2y}{dx^2}\right)^2-\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}\right]^2},\\
2\beta &= \frac{-3\dfrac{d^2y}{dx^2}\left\{\left[3\left(\dfrac{d^2y}{dx^2}\right)^2-\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}\right]\left[\left(\dfrac{dy}{dx}\right)^2-1\right]-2\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}\right\}}{\left(\dfrac{d^3y}{dx^3}\right)^2+\left[3\left(\dfrac{d^2y}{dx^2}\right)^2-\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}\right]^2}.
\end{aligned}\right.
$$

With the aid of (22) we can express these coördinates in terms of v', v'', v'''. We are considering the ∞^1 trajectories with the fixed initial element (o, o, v') but for which v'' and v''' are variables. Equations (23) and the equation of the trajectories (11) furnish us three equations from which we may eliminate v'' and v''', and thus get the equation of the focal locus. After considerable reductions we find for the equation of this locus

$$(24)\quad A_0(\alpha^2+\beta^2)^2+A_1(\alpha v'-\beta)(\alpha^2+\beta^2)+(\alpha v'-\beta)(A_2\alpha+A_3\beta)=0,$$

where A_0, A_1, A_2, A_3 are functions of u, v, v', the coördinates of the fixed element, only, and may thus be considered as constants. We find

$$
(25)\qquad \left\{\begin{aligned}
A_2 &= -3x_u^2(1+v'^2)(\psi v'^2+2\phi v'-\psi),\\
A_3 &= -3x_u^2(1+v'^2)(\phi v'^2-2\psi v'-\phi).
\end{aligned}\right.
$$

Now (24) is the equation of a bicircular quartic (a quartic possessing a pair of nodes at the circular points at infinity). The form of the equation indicates that the bicircular quartic (24) has the following properties:

(*i*) Its third double point is at the origin or initial point O and is a node, having two real and distinct tangents at O, viz.,

$$(26)\qquad \alpha v'-\beta=0 \quad\text{and}\quad A_2\alpha+A_3\beta=0,$$

the first of which has the direction of the initial element v'.

(*ii*) The inverse of the quartic with respect to O is

$$(27)\qquad (\alpha v'-\beta)(A_2\alpha+A_3\beta+A_1)+A_0=0$$

a hyperbola with $\alpha v'-\beta=0$, the initial element, as asymptote.[5]

[5] For a study of the bicircular quartic, see Basset, *Elementary treatise on cubic and quartic curves*, chapt. IX; also Loria, *Specielle algebraische und transscendente ebene Kurven*, pp. 102–108. It may easily be shown that property (*ii*) may be replaced by the following: the fundamental point of the bicircular quartic lies on a line through O perpendicular to the initial element.

We easily see that the equation of a bicircular quartic having properties (i) and (ii) is of the form (24). Hence we may state

THEOREM 2. *The ∞^1 trajectories passing through a given point in a given direction have associated with them their orthogonal projections in the tangent plane to the surface at the given point. The locus of the foci of the osculating parabolas of the associate system is a bicircular quartic having the given point as node and the given direction both as tangent line and also as one of the asymptotes of the hyperbola which is the inverse of the quartic with respect to the given point.*

Returning to our bicircular quartic (24), we have already noted that the given point O is a node and that the two tangents at O are given by (26). The first of these has the direction of the initial element v', and the second has the direction ξ given by

$$\xi = -\frac{A_2}{A_3} = -\frac{\psi v'^2 + 2\phi v' - \psi}{\phi v'^2 - 2\psi v' - \phi}. \tag{28}$$

Hence

$$\frac{\psi - \phi v'}{\phi + \psi v'} = \frac{v' - \xi}{1 + \xi v'}, \tag{29}$$

or

$$\tan\theta_1 = \tan\theta_2 \tag{30}$$

where θ_1 is the angle between the initial direction v' and the direction of the force vector ψ/ϕ, and θ_2 is the angle between the direction of the second tangent ξ and the initial direction v'. Hence we have

THEOREM 3. *The focal locus described in Theorem 1 has two distinct tangents at the given point. The initial element, which has the direction of one of these tangents, bisects the angle between the force vector and the second tangent.*

§ 4. Curves with Properties I and II.

Theorems 2 and 3 express geometric properties of the system of ∞^3 trajectories on the surface. The question arises whether these properties are characteristic of the system, i.e. whether the system of trajectories is the only system enjoying these properties. To answer this, let us now find all the systems of ∞^3 curves on a surface which possess

Property I. The ∞^1 curves passing through a given point in a given direction have associated with them their orthogonal projections

in the tangent plane to the surface at the given point. The locus of the foci of the osculating parabolas of the associate system is a bicircular quartic with the given point as node, and the given direction both as tangent line and also as one of the asymptotes to the hyperbola which is the inverse of the quartic with respect to the given point.

Any triply infinite system of curves on a surface may be represented by a differential equation of the form

$$v''' = f(u, v, v', v''). \tag{31}$$

Using the same notation and same coördinate system as in §3, the equation of a bicircular quartic described in Property I has the form

$$B_0(\alpha^2 + \beta^2)^2 + B_1(\alpha v' - \beta)(\alpha^2 + \beta^2) + (\alpha v' - \beta)(B_2\alpha + B_3\beta) = 0 \tag{32}$$

where the B's are arbitrary functions of u, v, v'. If in (32) we substitute for α, β their values as given by (23), we find with the aid of (22) and after considerable reduction, that the differential equation (31) has the form

$$v''' = A + Bv'' + Cv''^2, \tag{33}$$

where A, B, C, are arbitrary functions of u, v, v'. It is evident that equation (33) is much more general than equation (11). Thus we may state

Theorem 4. *The most general triply infinite system of curves on a surface possessing Property I, is defined by a differential equation of the form* (33) *involving three arbitrary functions of* u, v, v'.

Let us now convert Theorem 3. We must here replace the direction of the force vector through each point by a fixed direction through each point, but which may vary from point to point. We are to find the most general system of ∞^3 curves possessing Property I and also

Property II. The focal locus or bicircular quartic associated with each element (u, v, v') by Property I, is such that the initial element, which has the direction of one of the tangents, bisects the angle between a fixed direction through the initial point and the other tangent.

Let the fixed direction be given by $\omega\ (u, v)$. The system of curves possessing Property I is defined by the differential equation (33). For these curves the focal locus corresponding to the element (u, v, v') has for its second tangent the line

$$\alpha[Cv'(1 + v'^2) + 3(1 - v'^2)] - \beta[C(1 + v'^2) - 6v'] = 0 \tag{34}$$

whose direction is given by

$$\kappa = \frac{Cv'(1 + v'^2) + 3(1 - v'^2)}{C(1 + v'^2) - 6v'}. \tag{35}$$

For the bisection property we must have

$$\frac{\omega - v'}{1 + \omega v'} = \frac{v' - \kappa}{1 + \kappa v'}, \tag{36}$$

from which we find

$$C = \frac{3}{v' - \omega} \tag{37}$$

so that C is no longer an arbitrary function of u, v, v'. We may now state

THEOREM 5. *The most general triply infinite system of curves on a surface possessing Properties I and II is defined by a differential equation of the form*

$$v''' = A + Bv'' + \frac{3}{v' - \omega} v''^2, \tag{38}$$

involving two arbitrary functions A and B of u, v, v', and one arbitrary function ω of u, v.

Since properties I and II do not characterize our system of trajectories we shall seek further properties.

§ 5. Hyperosculation and the Central Locus.

Consider a point O on the surface. The geodesic curvature and center of geodesic curvature of a curve through O are respectively the curvature and center of curvature of the orthogonal projection of the curve on the tangent plane to the surface at O. Using the same coördinate system as in §4, we find that the geodesic curvature is

$$\frac{1}{\rho} = \frac{G}{e^{\lambda}(1 + v'^2)^{\frac{3}{2}}} = \frac{v'' - (\lambda_v - \lambda_u v')(1 + v'^2)}{e^{\lambda}(1 + v'^2)^{\frac{3}{2}}} \tag{39}$$

and that the coördinates of the center of geodesic curvature are

$$\xi = -\frac{v'\rho}{\sqrt{1 + v'^2}}, \quad \eta = \frac{\rho}{\sqrt{1 + v'^2}}. \tag{40}$$

Now, for each trajectory c which passes through O we may draw the curve g which osculates c and which has constant geodesic curvature (that of c at O) throughout. We call g the *osculating geodesic circle* of c. The question arises: how many of the ∞^1 trajectories which pass through O in a given direction v', will hyperosculate (have 4-point

contact with) the corresponding geodesic circles? To answer this we need simply apply the condition

$$\frac{d\rho}{ds} = 0 \tag{41}$$

to the form (39); we get

$$(1 + v'^2)G' - G\{(\lambda_u + \lambda_v v')(1 + v'^2) + 3v'v''\} = 0. \tag{42}$$

Substituting in this the value of G' from (11), and solving for v'', we find

$$v'' = \frac{\{(\psi_u - \lambda_u\psi + 2\lambda_v\phi) + (\psi_v - \phi_u - \lambda_u\phi + \lambda_v\psi)v' - (\phi_v - \lambda_v\phi + 2\lambda_u\psi)v'^2\}\{1 + v'^2\}}{3(\phi + \psi v')} \tag{43}$$

where we have discarded the factor G whose vanishing leads to the geodesics; as has already been noted, these curves form part of every system of trajectories; for the geodesics ρ, ξ, η are infinite. In (43) we note that for a given value of u, v, v', there corresponds only one value of v'', so that we have

THEOREM 6. *Through every point and in every direction through that point on a surface, there passes one trajectory which hyperosculates its corresponding geodesic circle of curvature.*

If, now, we keep the point O fixed and vary the initial direction v', the center of geodesic curvature of the hyperosculating trajectories of Theorem 6, will describe a certain locus. We get the equation of this locus by eliminating ρ, v', v'' from (39), (40), and (43). From (39) and (40) we find

$$v' = -\frac{\xi}{\eta}; \quad v'' = \frac{(\lambda_u\xi + \lambda_v\eta + e^\lambda)(\xi^2 + \eta^2)}{\eta^3}, \tag{44}$$

and substituting these in (43), we get

$$\xi^2(\phi_v - \lambda_v\phi - \lambda_u\psi) + \xi\eta(\psi_v - \phi_u + 2\lambda_u\phi - 2\lambda_v\psi) - \eta^2(\psi_u - \lambda_u\psi - \lambda_v\phi) + 3e^\lambda(\phi\eta - \psi\xi) = 0, \tag{45}$$

a conic passing through the point O. The equation of the tangent to this conic at O is

$$\phi\eta - \psi\xi = 0, \tag{46}$$

whose direction is that of the force vector ψ/ϕ. Hence, we have

THEOREM 7. *The locus of the centers of geodesic curvature of the* ∞^1

hyperosculating trajectories which pass through any point on the surface, is a conic passing through the point in the direction of the force vector.

We shall call this conic the *central locus.*

§ 6. Curves with Properties I, II, and III.

We shall now find all triply infinite systems of curves on a surface possessing Property I and the following

Property III. Through every point on the surface and in every direction through that point there passes one curve of the system which hyperosculates its corresponding geodesic circle of curvature. The locus of the centers of geodesic curvature of the ∞^1 hyperosculating trajectories which pass through a point is a conic passing through the point in a fixed direction.

To find all such systems we shall evidently have to convert Theorems 2, 6, and 7, replacing the direction of the force vector at any point by a fixed direction $\omega\ (u, v)$ through that point. By Theorem 4, a triply infinite system of curves possessing Property I may be represented by a differential equation of the form

$$v''' = A + Bv'' + Cv''^2. \tag{33}$$

The condition for hyperosculation of the curve and the element (u, v, v') is given by

$$(1 + v'^2)G' - G\{(\lambda_u + \lambda_v v')(1 + v'^2) + 3v'v''\} = 0, \tag{42}$$

where G and G' are given by (9) and v''' is to be replaced by its value from (33). Now $G = 0$ must satisfy (42), since the geodesic with element (u, v, v') certainly hyperosculates (indeed coincides with) its corresponding geodesic circle of curvature. And if, excluding the geodesic, there is to be only one curve of the system which has the hyperosculation property, equation (42) must give only one value of v''. Now G is linear in v'', and G' is quadratic in v'', so that G' must have the form

$$G' = G(a + bv'') \tag{47}$$

where a and b are functions of u, v, v' only. Equation (47) is therefore a restriction on the forms of the quantities A, B, C appearing in (33). Introducing (47) into (42), we get for the value of v'' corresponding to the direction of hyperosculation,

$$v'' = \frac{[a - (\lambda_u + \lambda_v v')]\,[1 + v'^2]}{3v' - b\,(1 + v'^2)}. \tag{48}$$

We may take for the equation of the conic of Property III lying in the tangent plane at O and passing through O in the fixed direction ω,

$$\alpha_0\xi^2 + \alpha_1\xi\eta + \alpha_2\eta^2 + 3e^{\lambda}(\eta - \omega\xi) = 0, \tag{49}$$

where α_0, α_1, α_2, ω are functions of the coördinates (u, v) of O only, and ξ, η are the coördinates of the center of geodesic curvature referred to O as origin. Substituting the values of ξ, η as given by (40), we find

$$v'' = \frac{(\beta_0 + \beta_1 v' + \beta_2 v'^2)(1 + v'^2)}{3(1 + \omega v')}, \tag{50}$$

where the β's are functions of u, v only. Comparing this with (48) we see that a and b are particular functions of u, v, v' satisfying the condition

$$\frac{a - (\lambda_u + \lambda_v v')}{3v' - b(1 + v'^2)} = \frac{(\beta_0 + \beta_1 v' + \beta_2 v'^2)}{3(1 + \omega v')}, \tag{51}$$

Finally, combining (47) and (51), we may state

THEOREM 8. *The most general triply infinite system of curves on a surface possessing Properties I and III, is defined by a differential equation of the form* (47), *in which*

$$a = \frac{[\beta_0 + \beta_1 v' + \beta_2 v'^2][3v' - b(1 + v'^2)]}{3(1 + \omega v')} + (\lambda_u + \lambda_v v').$$

The differential equation involves one arbitrary function of u, v, v' and four arbitrary functions of u, v.

Equation (47) expanded takes the form

$$v''' = D_0 + D_1 v'' + b v''^2 \tag{52}$$

where D_0 and D_1 are special functions of u, v, v'. From Theorem 5 we infer that all systems with Properties I and III will also possess Property II provided the function b has the form

$$b = \frac{3}{v' - \omega}. \tag{53}$$

Substituting this in (51), we find

$$a = -\frac{\gamma_0 + \gamma_1 v' + \gamma_2 v'^2}{v' - \omega}, \tag{54}$$

where the γ's are arbitrary functions of u, v. The value of v'' in (48) corresponding to a hyperosculating curve is now given by

$$(55)\quad v'' = \frac{\{(\gamma_0 + \gamma_1 v' + \gamma_2 v'^2) + (\lambda_u + \lambda_v v')(v' - \omega)\}\{1 + v'^2\}}{3(1 + \omega v')}.$$

We may finally state

THEOREM 9. *The most general triply infinite system of curves on a surface possessing Properties I, II, and III is defined by a differential equation of the form*

$$(56)\qquad (\omega - v')G' = G(\gamma_0 + \gamma_1 v' + \gamma_2 v'^2 - 3v'')$$

involving four arbitrary functions $\gamma_0, \gamma_1, \gamma_2, \omega$ *of* u, v.

By comparison of (56) with the differential equation of the trajectories

$$(11)\quad (\psi - \phi v')G' = G\{(\psi_u + 2\lambda_v\phi) + (\psi_v - \phi_u + 2\lambda_v\psi - 2\lambda_u\phi)v' - (\phi_v + 2\lambda_u\psi)v'^2 - 3\phi v''\},$$

involving only two arbitrary functions of u, v, we note that Properties I, II, and III are not sufficient to characterize the system of trajectories. We may here note the similarity in form of equations (11) and (56).

§ 7. The Lines of Force. Curves with Property I, II, III, IV.

On the surface, a line of force is a curve such that its tangent line at any point has the direction of the force vector through that point. The lines of force thus form a simple system of ∞^1 curves defined by the differential equation

$$(57)\qquad v' = \psi/\phi.$$

Employing (39), we find for the geodesic curvature of the line of force passing through the point O,

$$(58)\quad \frac{1}{\rho} = \frac{\phi^2\psi_u - \psi^2\phi_v + \phi\psi(\psi_v - \phi_u) - (\lambda_v\phi - \lambda_u\psi)(\phi^2 + \psi^2)}{e^\lambda(\phi^2 + \psi^2)^{\frac{3}{2}}}.$$

How does this compare with the curvature of the unique hyperosculating trajectory passing through O in the direction of the force vector? The value of v'' corresponding to a hyperosculating trajectory is given by (43); introducing this and the direction $v' = \psi/\phi$ into (39), we find for the required geodesic curvature,

$$(59)\quad \frac{1}{R} = \frac{\phi^2\psi_u - \psi^2\phi_v + \phi\psi(\psi_v - \phi_u) - (\lambda_v\phi - \lambda_u\psi)(\phi^2 + \psi^2)}{3e^\lambda(\phi^2 + \psi^2)^{\frac{3}{2}}}$$

Comparing (58) and (59), we may state

THEOREM 10. *For any point on the surface, the geodesic curvature of the line of force is equal to three times the geodesic curvature of the hyperosculating trajectory which passes out in the direction of the force vector.*

Let us now find the systems possessing Properties I, II, III, with the additional property got by converting Theorem 10. We replace the force vector by the fixed direction ω, tangent to the central locus of Property III. We may now ask for all the triply infinite systems of curves on a surface which possess Properties I, II, III, and

Property IV. With each point on the surface O, Property III associates a direction through the point, viz., the tangent to the central locus or conic. The totality of all such directions on the surface, defines a simple system of ∞^1 curves, which may be called the tangential lines. The geodesic curvature of the tangential line through O is equal to three times the geodesic curvature of the hyperosculating trajectory which passes through O in the same direction.

The geodesic curvature of the tangential line $v' = \omega$ through O is

$$\text{(60)} \qquad \frac{1}{\rho} = \frac{(\omega_u + \omega\omega_v) - (\lambda_v - \omega\lambda_u)(1 + \omega^2)}{e^{\lambda}(1 + \omega^2)^{\frac{3}{2}}}.$$

The curves possessing Properties I, II, III are defined by the differential equation (56). For the hyperosculating trajectory in the direction $v' = \omega$, we have, by (55).

$$v'' = \frac{\gamma_0 + \gamma_1\omega + \gamma_2\omega^2}{3},$$

and for the geodesic curvature,

$$\text{(61)} \qquad \frac{1}{R} = \frac{(\gamma_0 + \gamma_1\omega + \gamma_2\omega^2) - 3(\lambda_v - \omega\lambda_u)(1 + \omega^2)}{3e^{\lambda}(1 + \omega^2)^{\frac{3}{2}}}.$$

Setting $1/\rho$ equal to three times $1/R$, we find

$$\text{(62)} \qquad \gamma_0 + \gamma_1\omega + \gamma_2\omega^2 = (\omega_u + \omega\omega_v) + 2(\lambda_v - \omega\lambda_u)(1 + \omega^2).$$

Hence we have

THEOREM 11. *The most general triply infinite system of curves possessing Properties I, II, III, IV, is defined by a differential equation of the form (56) together with the condition (62), thus involving three arbitrary functions of u, v.*

We must therefore seek one other geometric property which would reduce the number of arbitrary functions of u, v in (56) to two and thus reduce this equation to that of the trajectories (11); this fifth property would then complete the characterization.

§ 8. Curves with Properties I, II, III, IV, V. — Complete Characterization.

The analytical expression for the final property is most readily found by comparing the coefficients in equations (56) and (11). If (56) is to reduce to (11), we must evidently have

$$(63)\quad \omega = \frac{\psi}{\phi};\quad \gamma_0 = \frac{\psi_u}{\phi} + 2\lambda_v;\quad \gamma_1 = \frac{\psi_v}{\phi} - \frac{\phi_u}{\phi} + 2\lambda_v \frac{\psi}{\phi} - 2\lambda_u;$$
$$\gamma_2 = -\frac{\phi_v}{\phi} - 2\lambda_u \frac{\psi}{\phi}.$$

Substituting

$$\psi = \omega\phi;\quad \psi_u = \omega_u\phi + \omega\phi_u;\quad \psi_v = \omega_v\phi + \omega\phi_v,$$

we get from the second and fourth of equations (63),

$$(64)\quad \frac{\phi_u}{\phi} = \frac{\gamma_0 - 2\lambda_v - \omega_u}{\omega} = (\log\phi)_u;\quad \frac{\phi_v}{\phi} = -\gamma_2 - 2\lambda_u\omega = (\log\phi)_v.$$

Substituting these values in the third equation (63), we have

$$(65)\quad \gamma_0 + \gamma_1\omega + \gamma_2\omega^2 = (\omega_u + \omega\omega_v) + 2(\lambda_v - \omega\lambda_u)(1 + \omega^2).$$

Equations (64) may be combined into

$$(66)\quad (\gamma_2 + 2\lambda_u\omega)_u + \left(\frac{\gamma_0 - 2\lambda_v - \omega_u}{\omega}\right)_v = 0.$$

If, then, (56) is to reduce to (11), the functions γ_0, γ_1, γ_2, ω must necessarily satisfy the relations (65) and (66). Conversely, if γ_0, γ_1, γ_2, ω satisfy (65) and (66), it is possible, by virtue of (66), to find a function $\log\phi$ (and hence ϕ) to satisfy both equations (64); and if we then choose $\psi = \omega\phi$, we shall have found a pair of functions ϕ, ψ, or a field of force, which satisfies all the equations (63). We may thus state

THEOREM 12. *In order that an equation of the form* (56) *should represent a system of trajectories under some field of force, it is necessary and sufficient that the four arbitrary functions of u, v satisfy equations* (65) *and* (66).

Now (65) is the same condition as (62), and we have already interpreted this geometrically by Property IV. It remains therefore to interpret condition (66) geometrically and thus complete the characterization.

Consider, at a point O, the isothermal u and v parameter curves and the hyperosculating curves of the system in these directions. Noting that $v'' = 0$ for the isothermal curves, we have for the geodesic curvatures of the u and v parameter curves,

$$\frac{1}{\rho_1} = -\frac{\lambda_u}{e^\lambda}; \quad \frac{1}{\rho_2} = -\frac{\lambda_v}{e^\lambda}.$$

Again, for the hyperosculating curves, v'' is given by (55), and the geodesic curvatures of these curves in the directions of the parameter curves are

$$\frac{1}{R_1} = \frac{\gamma_2 + \lambda_v + 3\lambda_u\omega}{-3e^\lambda\omega}; \quad \frac{1}{R_2} = \frac{\gamma_0 - \lambda_u\omega - 3\lambda_v}{3e^\lambda}.$$

Now (66) may be written

$$(67) \quad (\gamma_2 + \lambda_v + 3\lambda_u\omega)_u - (\lambda_u\omega)_u + \left(\frac{\gamma_0 - 3\lambda_v - \lambda_u\omega}{\omega}\right)_v + \left(\frac{\lambda_v}{\omega}\right)_v - (\log \omega)_{uv} = 0.$$

Introducing the values of ρ_1, ρ_2, R_1, R_2, this becomes

$$(68) \quad \left[e^\lambda\omega\left(\frac{1}{\rho_1} - \frac{3}{R_1}\right)\right]_u - \left[\frac{e^\lambda}{\omega}\left(\frac{1}{\rho_2} - \frac{3}{R_2}\right)\right]_v - (\log \omega)_{uv} = 0.$$

Introducing the abbreviations

$$(69) \quad \frac{1}{\kappa_1} = \omega\left(\frac{1}{\rho_1} - \frac{3}{R_1}\right); \quad \frac{1}{\kappa_2} = \frac{1}{\omega}\left(\frac{1}{\rho_2} - \frac{3}{R_2}\right),$$

and expanding (68), we get

$$(70) \quad e^\lambda\left[\left(\frac{1}{\kappa_1}\right)_u - \left(\frac{1}{\kappa_2}\right)_v\right] + e^\lambda\left[\frac{\lambda_u}{\kappa_1} - \frac{\lambda_v}{\kappa_2}\right] - [\log \omega]_{uv} = 0.$$

Finally, expressing λ_u and λ_v in terms of ρ_1 and ρ_2, dividing by $e^{2\lambda}$, and remembering that the arc lengths along the u and v isothermal parameter curves are given by

$$ds_1 = e^\lambda dv, \quad ds_2 = e^\lambda du$$

we may write (70) in the form

$$(71) \quad \frac{\partial}{\partial s_2}\left(\frac{1}{\kappa_1}\right) - \frac{\partial}{\partial s_1}\left(\frac{1}{\kappa_2}\right) - \frac{1}{\rho_1\kappa_1} + \frac{1}{\rho_2\kappa_2} - \frac{\partial^2(\log \omega)}{\partial s_1\, \partial s_2} = 0.$$

The quantities ρ_1, ρ_2, R_1, R_2, ω entering (71) are all geometric quantities, and (71) expresses a relation connecting their rates of variation

with respect to the arcs of the isothermal parameter curves as we move out on the surface from O. Furthermore, although the parameter curves seem to enter this relation, (71) is really an *intrinsic* property of our system, for it is evidently true for any and every set of orthogonal isothermal curves that may be chosen. We may now state

Property V. Construct any isothermal net on the surface. At any point O this net determines two orthogonal directions in which there pass two isothermal curves of the net and two hyperosculating curves of Property III. If ρ_1, ρ_2, R_1, R_2 are the radii of geodesic curvature of these four curves, s_1, s_2, the arc lengths along the isothermal curves, and ω, the tangent of the angle between the fixed direction of Property III and the isothermal curve with arc s_2, then, as we move along the surface from O, these quantities vary so as to satisfy the relation

$$\frac{\partial}{\partial s_2}\left(\frac{1}{\kappa_1}\right) - \frac{\partial}{\partial s_1}\left(\frac{1}{\kappa_2}\right) - \frac{1}{\rho_1\kappa_1} + \frac{1}{\rho_2\kappa_2} - \frac{\partial^2(\log \omega)}{\partial s_1\,\partial s_2} = 0,$$

where

$$\frac{1}{\kappa_1} = \omega\left(\frac{1}{\rho_1} - \frac{3}{R_1}\right), \quad \frac{1}{\kappa_2} = \frac{1}{\omega}\left(\frac{1}{\rho_2} - \frac{3}{R_2}\right).$$

Property V thus completes the characterization. We may now state

THEOREM 13. *In order that a triply infinite system of curves (∞^1 in each direction through each point) on a surface may be identified with a system of dynamical trajectories under any positional field of force, the given system must possess Properties I, II, III, IV, V.*

§ 9. Special Case.— Conservative Forces.

If the field of force is conservative, there exists a work function (negative potential) W of which the force components ϕ, ψ are the derivatives: hence

$$\phi = W_u, \quad \psi = W_v; \quad \text{or} \quad \psi_u = \phi_v. \tag{72}$$

We may interpret this relation geometrically by noting that if the conic (45) is to be a rectangular hyperbola, the sum of the coefficients of ξ^2 and η^2 must be zero; hence $\psi_u = \phi_v$; and conversely. Therefore, we have

THEOREM 14. *If the field of force is conservative, the locus of the*

centers of geodesic curvature of the ∞^1 *hyperosculating trajectories which pass through any point of the surface, is a rectangular hyperbola.*

Combining Theorems 13 and 14, we state

THEOREM 15. *In order that a triply infinite system of curves on a surface may be identified with a system of dynamical trajectories under a conservative field of force, the given system must possess Properties I, II, III, IV, V, and the additional property that the central locus of Property III is a rectangular hyperbola.*

§ 10. Brachistochrones, Catenaries, Dynamical Trajectories and Velocity Curves.

If the field of force is conservative, we may study certain types of ∞^3 curves on a surface other than dynamical trajectories. Among these, two cases of special interest are the systems of *brachistochrones* and *catenaries*. To get the equations of these systems we proceed as follows.

Consider the motion of a particle in a conservative field of force from one position P_0 to another P_1, with the sum of its kinetic and potential energies equal to a given constant. If T is the kinetic energy, W, the work function (negative potential), v, the velocity, and h, the constant of energy, we have

$$T - W = h, \text{ or } \tfrac{1}{2}v^2 - W = h, \text{ or } v^2 = 2(W + h). \tag{73}$$

(*i*) If the motion takes place under the *principle of least action*, i.e., so that

(74) *Action* =

$$\int_{(P_0)}^{(P_1)} 2T\,dt = \int_{(P_0)}^{(P_1)} v^2 dt = \int_{(P_0)}^{(P_1)} v\,ds = \int_{(P_0)}^{(P_1)} \sqrt{2(W+h)}\,ds = \textit{minimum},$$

the paths are *dynamical trajectories*.

(*ii*) If the motion takes place so that the time elapsed is least, i.e., so that

$$\textit{Time} = \int_{(P_0)}^{(P_1)} dt = \int_{(P_0)}^{(P_1)} \frac{ds}{v} = \int_{(P_0)}^{(P_1)} \frac{ds}{\sqrt{2(W+h)}} = \textit{minimum}, \tag{75}$$

the paths are *brachistochrones*.

(*iii*) If the motion takes place along the position of equilibrium of a homogeneous flexible inextensible string, then

$$(76) \qquad \int_{(P_0)}^{(P_1)} v^2\, ds = \int_{(P_0)}^{(P_1)} 2(W + h)ds = minimum,$$

and the paths are *catenaries*.

For a given constant of energy, h, (74), (75), or (76) will give ∞^2 curves, one through each point in each direction on the surface.[7] If we allow h to vary, we shall get triply infinite systems of curves: complete systems of dynamical trajectories, brachistochrones, or catenaries. The systems defined by (74), (75), and (76) may be considered as special cases of the system defined by

$$(77) \qquad \int_{(P_0)}^{(P_1)} (W + h)^{\frac{m}{2}}\, ds = minimum,$$

where we have trajectories, brachistochrones, or catenaries, according as $m = 1, -1$, or 2.

Replacing ds by $e^{\lambda}\sqrt{1 + v'^2}\, du$, and applying the Euler condition for the vanishing of the first variation, to

$$(78) \qquad \int (W + h)^{\frac{m}{2}} e^{\lambda} \sqrt{1 + v'^2}\, du = \int H\, du = minimum,$$

viz.,

$$H_v - H_{v'u} - v' H_{v'v} - v'' H_{v'v'} = 0,$$

we find

$$(79) \quad v'' = \left\{ \left[log\, (W+h)^{\frac{m}{2}} + \lambda \right]_v - \left[log\, (W+h)^{\frac{m}{2}} + \lambda \right]_u v' \right\} \left\{ 1 + v'^2 \right\},$$

as the differential equation of the system of ∞^2 curves. To find the differential equation of the system of ∞^3 curves, we must differentiate (79) and eliminate h. This is most readily done by writing (79) in the form

$$\frac{m}{2}\, \frac{W_v - W_u v'}{W + h} = \frac{v'' - (\lambda_v - \lambda_u v')\,(1 + v'^2)}{1 + v'^2} = \frac{G}{1 + v'^2},$$

---

7 These systems of ∞^2 curves form special cases of the extremals connected with a variation problem of the form $\int F ds$ = minimum, where F is a function of the coördinates. Such systems, termed "*Natural Systems*," have been characterized geometrically by the author in "*Natural families of curves in a general curved space of N dimensions*," Trans. Am. Math. Soc., vol. 13 (1912), pp. 77–95. The author has also characterized these curves in a different way in "*Some geometric investigations on the general problem of dynamics*," Proceedings of the Am. Academy of Arts and Sciences, Vol. 55 (1920), pp. 285–322.

where G is defined as in (9), or

$$\frac{2}{m}(W+h) = \frac{(W_v - W_u v')(1+v'^2)}{G}.$$

Differentiating this last expression, we get

$$(80) \quad \frac{2}{m}(W_u + W_v v') = \frac{G\dfrac{d}{du}\left\{(W_v - W_u v')(1+v'^2)\right\} - G'(W_v - W_u v')(1+v'^2)}{G^2}.$$

If we introduce the components of the force

$$(81) \qquad \phi = W_u, \quad \psi = W_v,$$

and solve for G', at the same time replacing $2/m$ by n, we find

$$(82) \quad (\psi - \phi v')G' = G\left\{(\psi_u + n\lambda_v\phi) + (\psi_v - \phi_u + n\lambda_v\psi - n\lambda_u\phi)v' - (\phi_v + n\lambda_u\psi)v'^2 + \left[(2-n)\frac{\phi + \psi v'}{1+v'^2} - 3\phi\right]v''\right\}$$

for the required differential equation, where

$n = 2$ corresponds to dynamical trajectories
$n = -2$ " " brachistochrones
$n = 1$ " " catenaries,

and G and G' are defined by (9). We shall designate the curves defined by (82) as an "n" system. We may here note the similarity of equation (82) for the "n" system and equation (11) for the dynamical trajectories in any field of force.

We may also study certain other types of curves on a surface, termed "velocity" curves. They are defined dynamically as follows: A curve is a velocity curve corresponding to the speed $\dot{s}_0$, if a particle starting with that speed from any point of such a curve and in the direction of the curve, describes a trajectory osculating the curve.

To get the differential equation of such a system, we note that the differential equation (11) of the trajectories for any positional field of force was obtained by eliminating the variable component $\dot{u}$ of the velocity from equation (8). Using the relation

$$\dot{s}^2 = e^{2\lambda}(\dot{u}^2 + \dot{v}^2) = e^{2\lambda}\dot{u}^2(1+v'^2),$$

we may write equation (8) in the form

$$(8') \qquad v'' = \frac{1}{\dot{s}^2}[(\psi + \lambda_v) - (\phi + \lambda_u)v'][1 + v'^2].$$

Equation (8′) holds for any trajectory and along this the velocity $\dot{s}$ varies from point to point. Now, if in (8′) we replace $1/\dot{s}^2$ by a constant c, we get

$$(8'') \qquad v'' = c[(\psi + \lambda_v) - (\phi + \lambda_u)v'][1 + v'^2],$$

a differential equation of the second order representing a system of ∞^2 curves, one through each point in each direction. Each of these curves, therefore, has the dynamical property used above in defining a velocity curve.[8]

For each constant value assigned to the speed $\dot{s}$, we get a velocity system, and the totality of ∞^1 systems obtained by varying $\dot{s}$ constitute a complete velocity system of ∞^3 curves on the surface. To find the differential equation of the complete velocity system, we, therefore, differentiate (8″) and eliminate the parameter c. Writing (8″) in the form

$$\frac{1}{c} = \frac{(\psi - \phi v')(1 + v'^2)}{G},$$

we get by direct differentiation,

$$(82') \quad (\psi - \phi v')G' = G\left\{\left[\psi_u + (\psi_v - \phi_u)v' - \phi_v v'^2\right] + \left[2\frac{\phi + \psi v'}{1 + v'^2} - 3\phi\right]v''\right\},$$

for the required differential equation of a complete velocity system. We now note that this is the form taken by equation (82) if $n = 0$. Hence, we may state that *an "n" system represents a velocity system when $n = 0$.* For a velocity system in a conservative field of force we merely add conditions (81).

§ 11. Geometric Characterization of "n" Systems.

Let us now find the geometric properties of the system defined by (82). Since this equation has the form of equation

8 A velocity system of ∞^2 curves is characterized geometrically by the fact that the locus of the centers of geodesic curvature of the ∞^1 curves which pass through a given point, is a straight line. See the author's paper "*Geometric characterization of isogonal trajectories on a surface,*" Annals of Math., 2d series, **15** (1913), pp. 71–77. For further discussion of velocity systems see the author's paper "*Note on velocity systems in curved spece of n-dimensions,*" Bull. Am. Math. Soc. 2d series, **27** (1920), pp. 71–77.

(33) $$v''' = A + Bv'' + Cv''^2,$$

Theorem 4 is applicable here, hence we have

THEOREM 16. *The "n" system possesses Property I.*

For the "n" system,

(83) $$C = \frac{1}{\psi - \phi v'}\left[(2-n)\frac{\phi + \psi v'}{1 + v'^2} - 3\phi\right];$$

hence the bisection property II does not hold unless $n = 2$, i.e., for dynamical trajectories. For the general "n" system the bicircular quartic of Property I has a node at the given point, one of the tangents having the direction of the initial element v', and the other having the direction.

(84) $$\xi = \frac{Cv'(1 + v'^2) + 3(1 - v'^2)}{C(1 + v'^2) - 6v'},$$

where C is given by (83). Substituting this value and introducing the direction of the force vector $\psi/\phi = \omega$, we may write (84) as

(85) $$\frac{\omega - v'}{1 + \omega v'} = \frac{n+1}{3}\,\frac{v' - \xi}{1 + v'\xi},$$

or

(86) $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{n+1}{3},$$

where θ_1 is the angle between the initial direction and the force vector, and θ_2 is the angle between the direction of the second tangent and the initial direction. Conversely, we may easily show that if the angles θ_1 and θ_2 are related as in (86), C must have the value given in (83). We thus have

THEOREM 17. **Property II′.** *For an "n" system, the focal locus described by Property I has two distinct tangent lines at the initial point. The tangent of the angle which the initial element makes with the force vector is to the tangent of the angle which the second tangent line makes with the initial element as $n + 1$ is to 3.*

We may easily show that the most general triply infinite system of curves possessing Properties I and II′ has an equation of the form

$$v''' = A + Bv'' + \frac{1}{\omega - v'}\left\{(2-n)\frac{1 + \omega v'}{1 + v'^2} - 3\right\}v''^2$$

where a fixed direction $\omega\,(u, v)$ replaces the direction of the force vector.

We further find that through every point and in every direction through that point there passes one trajectory which hyperosculates its corresponding geodesic circle of curvature. This trajectory is given by

$$(87)\quad \frac{v''}{1+v'^2} = \frac{[\psi_u-\lambda_u\psi+n\lambda_v\phi]+[\psi_v-\phi_u+(1-n)(\lambda_u\phi-\lambda_v\psi)]v'-[\phi_v-\lambda_v\phi+n\lambda_u\psi]v'^2}{(1+n)(\phi+\psi v')},$$

and the central locus, or locus of centers of geodesic curvature of the ∞^1 hyperosculating trajectories which pass through any point on the surface, is a conic passing through the given point in the direction of the force vector. The equation of this conic is

$$(88)\quad \xi^2(\phi_v-\lambda_v\phi-\lambda_u\psi)+\xi\eta(\psi_v-\phi_u+2\lambda_u\phi-2\lambda_v\psi) \\ -\eta^2(\psi_u-\lambda_u\psi-\lambda_v\phi)+(1+n)e^\lambda(\phi\eta-\psi\xi)=0.$$

Since, by (81), $\phi_v = \psi_u$, this conic is a rectangular hyperbola. Hence

THEOREM 18. *The "n" system possesses Property III. The conic described in this property is a rectangular hyperbola.*

We may now show that the most general triply infinite system of curves possessing Properties I, II′, III, has an equation of the form

$$(89)\quad (\omega-v')\,G' = G\left\{\gamma_0+\gamma_1 v'+\gamma_2 v'^2+\left[(2-n)\frac{1+\omega v'}{1+v'^2}-3\right]v''\right\},$$

where $\gamma_0, \gamma_1, \gamma_2, \omega$ are arbitrary functions of u, v.

From (88) we conclude that at any point O the asymptotes of the rectangular hyperbolas associated with all "n" systems are parallel, and that the locus of the centers of these hyperbolas is a straight line through O.

We further find that the geodesic curvature of the hyperosculating curve (87) which passes out in the direction of the force vector ψ/ϕ is

$$(90)\quad \frac{1}{R} = \frac{\phi^2\psi_u-\psi^2\phi_v+\phi\psi(\psi_v-\phi_u)-(\lambda_v\phi-\lambda_u\psi)(\phi^2+\psi^2)}{(1+n)\,e^\lambda\,(\phi^2+\psi^2)^{\frac{3}{2}}}.$$

Comparing this with the geodesic curvature $1/\rho$ of the line of force $v' = \psi/\phi$ given by (58), we conclude that

$$(91)\quad \frac{1}{\rho} = \frac{1+n}{R}.$$

Hence,

THEOREM 19. **Property IV′.** *For an "n" system, at any point on the surface, the geodesic curvature of the line of force is equal to* $(n+1)$

times the geodesic curvature of the hyperosculating curve which passes through the point in the direction of the force vector.

We may now show that the most general triply infinite system of curves possessing Properties I, II′, III, defined by the differential equation (89), will also possess Property IV′, where a fixed direction ω replaces the direction of the force vector, provided the condition

$$(92)\qquad \gamma_0 + \gamma_1\omega + \gamma_2\omega^2 = (\omega_u + \omega\omega_v) + n(\lambda_v - \lambda_u\omega)(1 + \omega^2)$$

is satisfied.

It is evident that Properties I, II′, III, IV′ do not completely characterize an "n" system. To complete the characterization, let us compare the differential equations (89) and (82); we evidently have

$$(93)\qquad \omega = \frac{\psi}{\phi};\quad \gamma_0 = \frac{\psi_u}{\phi} + n\lambda_v;\quad \gamma_1 = \frac{\psi_v}{\phi} - \frac{\phi_u}{\phi} + n\left(\lambda_v\frac{\psi}{\phi} - \lambda_u\right);$$

$$\gamma_2 = -\frac{\phi_v}{\phi} - n\lambda_u\frac{\psi}{\phi}.$$

As in §8, conditions (93) may be reduced to (92) and

$$(94)\qquad \left(\gamma_2 + n\lambda_u\omega\right)_u + \left(\frac{\gamma_0 - n\lambda_v - \omega_u}{\omega}\right)_v = 0.$$

In order that an equation of the form (89) should represent an "n" system, it is, therefore, necessary and sufficient that the four arbitrary functions γ_0, γ_1, γ_2, ω satisfy (92) and (94).

To interpret (94) geometrically, we may write it in the form

$$(95)\qquad \left[\gamma_2 + \lambda_v + (n+1)\lambda_u\omega\right]_u - \left[\lambda_u\omega\right]_u + \left[\frac{\gamma_0 - (n+1)\lambda_v - \lambda_u\omega}{\omega}\right]_v$$

$$+ \left[\frac{\lambda_v}{\omega}\right]_v - \left[\log\omega\right]_{uv} = 0.$$

Introducing here the geodesic curvatures

$$\frac{1}{\rho_1} = -\frac{\lambda_u}{e^{\lambda}},\quad \frac{1}{\rho_2} = -\frac{\lambda_v}{e^{\lambda}}$$

of the isothermal u and v parameter curves, and the geodesic curvatures

$$\frac{1}{R_1} = \frac{\gamma_2 + \lambda_v + (n+1)\lambda_u\omega}{-(1+n)e^{\lambda}\omega},\quad \frac{1}{R_2} = \frac{\gamma_0 - (n+1)\lambda_v - \lambda_u\omega}{(1+n)e^{\lambda}},$$

of the hyperosculating curves of Property III which have the directions of the u and v parameter curves, (95) becomes

$$(96)\quad \left[e^{\lambda}\omega\left(\frac{1}{\rho_1}-\frac{n+1}{R_1}\right)\right]_u-\left[\frac{e^{\lambda}}{\omega}\left(\frac{1}{\rho_2}-\frac{n+1}{R_2}\right)\right]_v-\left[\log\omega\right]_{uv}=0.$$

Introducing the abbreviations

$$(97)\qquad \frac{1}{\kappa_1}=\omega\left(\frac{1}{\rho_1}-\frac{n+1}{R_1}\right),\ \frac{1}{\kappa_2}=\frac{1}{\omega}\left(\frac{1}{\rho_2}-\frac{n+1}{R_2}\right),$$

and expanding (96), we get

$$(98)\qquad e^{\lambda}\left[\left(\frac{1}{\kappa_1}\right)_u-\left(\frac{1}{\kappa_2}\right)_v\right]+e^{\lambda}\left[\frac{\lambda_u}{\kappa_1}-\frac{\lambda_v}{\kappa_2}\right]-\left[\log\omega\right]_{uv}=0.$$

Finally, expressing λ_u and λ_v in terms of ρ_1 and ρ_2, dividing by $e^{2\lambda}$, and employing the arc lengths

$$ds_1=e^{\lambda}\,dv,\quad ds_2=e^{\lambda}\,du$$

along the u and v isothermal parameter curves, (98) becomes

$$(99)\qquad \frac{\partial}{\partial s_2}\left(\frac{1}{\kappa_1}\right)-\frac{\partial}{\partial s_1}\left(\frac{1}{\kappa_2}\right)-\frac{1}{\rho_1\kappa_1}+\frac{1}{\rho_2\kappa_2}-\frac{\partial^2(\log\omega)}{\partial s_1\,\partial s_2}=0.$$

The quantities ρ_1, ρ_2, R_1, R_2, ω in (99) are all geometric quantities, and this equation expresses an intrinsic property of our "n" system, for it is evidently true for any and every set of orthogonal isothermal curves that may be chosen. Hence,

THEOREM 20. **Property V′.** *Construct any isothermal net on the surface. At any point O, this net determines two orthogonal directions in which there pass two isothermal curves of the net and two hyperosculating curves of Property III. If ρ_1, ρ_2, R_1, R_2 are the radii of geodesic curvature of these four curves, s_1, s_2, the arc lengths along the isothermal curves, and ω, the tangent of the angle between the fixed direction of Property III and the isothermal curve with arc s_2, then, as we move along the surface from O, these quantities vary so as to satisfy the relation*

$$\frac{\partial}{\partial s_2}\left(\frac{1}{\kappa_1}\right)-\frac{\partial}{\partial s_1}\left(\frac{1}{\kappa_2}\right)-\frac{1}{\rho_1\kappa_1}+\frac{1}{\rho_2\kappa_2}-\frac{\partial^2(\log\omega)}{\partial s_1\,\partial s_2}=0,$$

where

$$\frac{1}{\kappa_1}=\omega\left(\frac{1}{\rho_1}-\frac{n+1}{R_1}\right),\ \frac{1}{\kappa_2}=\frac{1}{\omega}\left(\frac{1}{\rho_2}-\frac{n+1}{R_2}\right).$$

Property V′ thus completes the characterization of an "n" system, and we may finally state

THEOREM 21. *In order that a triply infinite system of curves on a surface may be identified with an "n" system, the given system must possess Properties I, II′, III, IV′, V′.*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
Cambridge, Mass., September, 1920.